

# GOLDBACH VERSUS DE POLIGNAC NUMBERS

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**ABSTRACT.** In this article we prove a second moment estimate for the Maynard-Tao sieve and give an application to Goldbach and de Polignac numbers. We show that at least one of two nice properties holds. Either consecutive Goldbach numbers lie within a finite distance from one another or else the set of de Polignac numbers has full density in  $2\mathbb{N}$ .

## 1. INTRODUCTION

Let  $\mathcal{P}$  denote the set of prime numbers and write  $p_n$  for its  $n$ -th member. Given an admissible tuple of integers  $\mathcal{H} = \{h_1, \dots, h_k\}$  the Hardy-Littlewood  $k$ -tuple prime conjecture is the assertion that

$$\{n + h_1, \dots, n + h_k\} \subset \mathcal{P}$$

for infinitely many integers  $n$ . The problem has seen a number of breakthroughs over the past decade and these efforts spawned an international collaboration known as the Polymath8 project ([2]). Assuming the generalised Elliott-Halberstam conjecture, it was demonstrated that any admissible configuration  $\{n + h_1, n + h_2, n + h_3\}$  contains at least two primes for infinitely many values of  $n$ . These ideas can be applied to Goldbach numbers, that is to say, positive integers which can be expressed as the sum of two primes. Fixing some large natural number  $N$  one considers the collection  $\{n, n + 2, N - n\}$  and in this manner it can be shown, under suitable hypotheses, that at least one of the following statements must hold

- (i) There are infinitely many twin primes.
- (ii) One has  $g_{n+1} - g_n \leq 4$  for all sufficiently large  $n$ .

Here  $g_n$  denotes the  $n$ -th Goldbach number. In this paper we prove a result of the same nature. To state the theorem, we say  $m$  is a de Polignac number if there exist infinitely many pairs of primes  $(p, p')$  such that  $p - p' = m$ . Let  $\mathcal{D}$  denote the set of de Polignac numbers.

**Theorem 1.1.** *At least one of the following statements must hold*

- (i) *There exists an absolute constant  $C > 0$  so that  $g_{n+1} - g_n \leq C$  for all sufficiently large  $n$ .*
- (ii) *The set  $\mathcal{D}$  has full asymptotic density in the even numbers and more precisely*

$$(1.1) \quad |\mathcal{D}^c \cap [0, N]| \leq N^\kappa$$

*for all large  $N$  and some  $\kappa < 1$ .*

We note that this result is unconditional while the Polymath theorem relies on the powerful Elliott-Halberstam Conjecture. It will, however, be necessary to push just beyond the reach of the Bombieri-Vinogradov theorem and make use of Zhang type equidistribution estimates. The proof of Theorem 1.1 employs a second moment estimate for the Maynard-Tao sieve weight (see [5]) together with a Cauchy-Schwarz argument. We will show that a similar argument can be applied to the sequence of normalised prime gaps. Letting  $\mathcal{L}$  denote the set of limit points for the

sequence  $(p_{n+1} - p_n)/\log p_n$ , Banks, Freiberg and Maynard [1] established the lower bounds

$$\liminf_{T \rightarrow \infty} \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{1}{8} \quad \text{and} \quad \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{1}{22} \quad \forall T > 0,$$

where  $m$  denotes the Lebesgue measure on  $\mathbb{R}$ . The asymptotic density estimate is ineffective in  $T$ . We give a simple extension of this result.

**Proposition 1.2.** *The limit set  $\mathcal{L}$  obeys the estimates*

$$\liminf_{T \rightarrow \infty} \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{1}{4} \quad \text{and} \quad \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{3}{25} \quad \forall T > 0,$$

with the first estimate being ineffective in  $T$ .

**Remark.** Assuming a variation on the Elliott-Halberstam conjecture we will prove in Section 5 that the constant  $1/4$  may be replaced with  $1/2$ .

**Notation** We introduce some standard notation that will be used throughout the paper. For functions  $f$  and  $g$  we will use the symbols  $f \ll g$  and  $f = O(g)$  interchangeably to express Landau's big O symbol. A subscript of the form  $\ll_\eta$  means the implied constant may depend on the quantity  $\eta$ . The statement  $f \sim g$  means  $f$  and  $g$  are asymptotically equivalent, i.e.,  $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$  and we will write  $r(N, k) = o_k(1)$  when  $\lim_{k \rightarrow \infty} r(N, k) = 0$ , independently of  $N$ . Given a natural number  $m$ , we write  $P^+(m)$  for its largest prime divisor. We reserve the letter  $\mu$  for the Möbius function and write  $[N] = \{1, 2, \dots, N\}$  for any natural number  $N$ .

## 2. SETTING UP THE SIEVE

**2.1. The general framework.** In order to obtain clusters of primes in bounded intervals one considers sums of the form

$$(2.1) \quad S = \sum_{n \leq N} \left( \sum_{i=1}^k 1_{\mathcal{P}}(n + h_i) - (m-1) \right) w(n)^2.$$

When  $S > 0$  we necessarily have some  $m$ -tuple  $(n + h_{i_1}, \dots, n + h_{i_m})$  consisting entirely of primes. The weight function  $w(n)$  takes the shape

$$(2.2) \quad w(n) = \sum_{\substack{d_1, \dots, d_k \\ d_i | n + h_i \forall i \\ \mathbf{d} \leq R}} \lambda_{\mathbf{d}}$$

where  $\mathbf{d} = (d_1, \dots, d_k)$  denotes a  $k$ -tuple of positive integers,  $\mathbf{d} = \prod_{i=1}^k d_i$  and

$$(2.3) \quad \lambda_{\mathbf{d}} = \left( \prod_{i=1}^k \mu(d_i) \right) f \left( \frac{\log d_1}{\log R}, \dots, \frac{\log d_k}{\log R} \right)$$

for some smooth function  $f : [0, \infty)^k \rightarrow \mathbb{R}$  supported on  $\Delta_k := \left\{ t_1, \dots, t_k \geq 0 \mid 0 \leq \sum_{i=1}^k t_i \leq 1 \right\}$ . Due to a technical restriction, which will be pointed out in the appendix, it is in fact necessary to reduce the size of the simplex. We define for any pair of real numbers  $0 \leq \eta < \tau < 1$ , the region  $\Delta_k(\eta, \tau) = \left\{ t_1, \dots, t_k \geq \eta \mid \sum_{i=1}^k t_i \leq \tau \right\}$ . The truncation parameter  $R = N^\delta$ , with  $0 < \delta < 1$ , depends on the level of distribution of the primes. We also let  $w := \log \log \log N$ , set  $W = \prod_{p \leq w} p$

and choose a residue class  $b_0 \bmod W$  with  $(b_0, W) = 1$ . With regards to the partial derivatives of  $f$ , define for each  $1 \leq j \leq k$ ,

$$Df = \frac{\partial^k}{\partial t_1 \dots \partial t_k} f \quad \text{and} \quad D_j f = \frac{\partial^{k-1}}{\partial t_1 \dots \partial t_{j-1} \partial t_{j+1} \dots \partial t_k} f.$$

We record here the asymptotic estimates required to compute the sums appearing in  $S$ . They are essentially proven in [2, Section 5] but we will give a short sketch of these results in the appendix.

**Proposition 2.1.** *Let  $k \in \mathbb{N}$  be sufficiently large. Under the assumptions and notation introduced above, there exist constants  $\delta > 1/4$  and  $\sigma > 0$  with the following property. For any smooth function  $f : [0, \infty)^k \rightarrow \mathbb{R}$  supported on  $\Delta_k(0, \sigma)$  and for each choice of index  $1 \leq i_0 \leq k$  one has the estimates*

$$(2.4) \quad \sum'_{n \leq N} 1_{\mathcal{P}}(n + h_{i_0}) w(n)^2 \sim \delta N \beta(N) J_k^{(i_0)}(f) \quad \text{and}$$

$$(2.5) \quad \sum'_{n \leq N} w(n)^2 \sim N \beta(N) I_k(f).$$

The superscript  $'$  indicates that  $n$  is made to run through natural numbers in the residue class  $b_0 \bmod W$ .  $J$  and  $I$  are integrals given by

$$(2.6) \quad J_k^{(l)}(f) = \int D_l f(t_1, \dots, t_{l-1}, 0, t_{l+1}, \dots, t_k)^2 dt_1 \dots dt_{l-1} dt_{l+1} \dots dt_k,$$

$$(2.7) \quad I_k(f) = \int Df(t_1, \dots, t_k)^2 dt_1 \dots dt_k$$

and

$$\beta(N) = \beta(N, W) = \frac{W^k}{\varphi(W)^k} (\log N)^k.$$

The main ingredient in the proof of Theorem 1.1 will be a second moment estimate for the weight  $\sum_{i=1}^k 1_{\mathcal{P}}(n + h_i) w(n)$ . Using this bound we will finish the argument in Section 4.

**Proposition 2.2** (Second moment estimate). *Let  $\psi : \mathbb{N} \rightarrow \mathbb{R}$  be a positive sequence tending to zero and suppose that  $\psi(k) \log k \rightarrow \infty$ . Under the assumptions outlined above, there exists a smooth function  $f : [0, \infty)^k \rightarrow \mathbb{R}$  supported on  $\Delta_k(0, \sigma)$  satisfying the estimates <sup>1</sup>*

$$(2.8) \quad \sum'_{n \leq N} 1_{\mathcal{P}}(n + h_{i_0}) w(n)^2 \sim \psi(k) \frac{\log k}{k} \delta N \beta(N) I_k(f) (1 + o_k(1)),$$

$$(2.9) \quad \sum'_{n \leq N} 1_{\mathcal{P}}(n + h_i) 1_{\mathcal{P}}(n + h_j) w(n)^2 \leq \delta \psi(k)^2 \frac{(\log k)^2}{k^2} N \beta(N) I_k(f) (1 + o_k(1))$$

for all  $h_{i_0}$  and all pairs  $h_i \neq h_j$  in  $\mathcal{H}$ .

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<sup>1</sup>Recall the notation  $r(N, k) = o_k(1)$  when  $\lim_{k \rightarrow \infty} r(N, k) = 0$ , independently of  $N$ .

## 3. A VARIATIONAL PROBLEM

We begin the proof of Proposition 2.2 with a nice observation made by T. Tao in the blog post [7]. Given  $h_i \neq h_j$ , one has that

$$1_{\mathcal{P}}(n + h_j)1_{\mathcal{P}}(n + h_i) \left( \sum_{\substack{d_1, \dots, d_k \\ d_r | n + h_r \\ \mathbf{d} \leq R}} \lambda_{\mathbf{d}} \right)^2 \leq 1_{\mathcal{P}}(n + h_i) \left( \sum_{\substack{d_1, \dots, d_k \\ d_r | n + h_r \\ \mathbf{d} \leq R}} \tilde{\lambda}_{\mathbf{d}} \right)^2$$

provided that  $\tilde{\lambda}$  satisfies  $\tilde{\lambda}_{\mathbf{d}} = \lambda_{\mathbf{d}}$  whenever  $d_i = d_j = 1$ . Since the values of  $i$  and  $j$  will have no bearing on the argument, let us assume for notational convenience that  $i = 1$  and  $j = k$ . Defining  $\tilde{\lambda}$  as in (2.3), we have thus reduced our problem to that of finding a function  $\tilde{f}$  which minimises

$$\tilde{M}_k := \frac{\sum'_{n \leq N} 1_{\mathcal{P}}(n + h_1) \tilde{w}(n)^2}{\sum'_{n \leq N} w(n)^2}$$

subject to the condition

$$(3.1) \quad \tilde{f}(0, t_2, \dots, t_{k-1}, 0) = f(0, t_2, \dots, t_{k-1}, 0).$$

An application of Proposition 2.1 gives the asymptotic

$$\tilde{M}_k \sim \delta \frac{\int_{\Delta_{k-1}} D_k \tilde{f}(t_1, t_2, \dots, t_{k-1}, 0)^2 dt_1 \dots dt_{k-1}}{\int_{\Delta_k} Df(t_1, t_2, \dots, t_k)^2 dt_1 \dots dt_k}.$$

Using the techniques developed in [2] it can be shown that for some specific choice of  $f$  one has the bound

$$(3.2) \quad \frac{\int_{\Delta_{k-1}} D_k f(t_1, t_2, \dots, t_{k-1}, 0)^2 dt_1 \dots dt_{k-1}}{\int_{\Delta_k} Df(t_1, t_2, \dots, t_k)^2 dt_1 \dots dt_k} \sim \psi(k) \frac{\log k}{k} (1 + o_k(1)).$$

We will revisit this estimate in the next section but let us assume for the time being that (3.2) holds. Then it remains to minimise  $\int D_k \tilde{f}(t_1, t_2, \dots, t_{k-1}, 0)$  under the constraint (3.1). By the Euler-Lagrange equations, the extremiser  $\tilde{f}$  must satisfy  $\frac{\partial}{\partial t_1} D_k \tilde{f} = 0$ . Applying the boundary conditions  $f(\partial\Delta) = 0$  together with (3.1) one finds the minimiser  $\tilde{f}$  for which

$$D_k \tilde{f}(t_1, t_2, \dots, t_{k-1}, 0) = - \left[ \frac{\partial^{k-2}}{\partial t_2 \dots \partial t_{k-1}} f(0, t_2, \dots, t_{k-1}, 0) \right] / (1 - t_2 - \dots - t_{k-1}).$$

To avoid issues on the boundary  $\partial\Delta_k$  recall that we defined  $\Delta_k(\eta, \tau) = \left\{ t_1, \dots, t_k \geq \eta \mid \sum_{i=1}^k t_i \leq \tau \right\}$  for any pair of real numbers  $0 \leq \eta < \tau < 1$ . Introducing the notation  $F(t_1, \dots, t_k) = \frac{\partial f}{\partial t_1 \dots \partial t_k}$  and applying the fundamental theorem of calculus, it remains to choose a  $0 < \tau < 1$  and bound the

ratio

$$\begin{aligned}
& \frac{\int \left( \int \int F(t_1, \dots, t_k) dt_1 \dots dt_k \right)^2 / (1 - \sum_{i=2}^{k-1} t_i)^2 dt_2 \dots dt_{k-1}}{\int \left( \int F(t_1, \dots, t_k) dt_1 \right)^2 dt_2 \dots dt_k} \\
& \leq \frac{\int_{\Delta_k(0, \tau)} \left( \int \int F(t_1, \dots, t_k) dt_1 \dots dt_k \right)^2 / (1 - \sum_{i=2}^{k-1} t_i)^2 dt_2 \dots dt_{k-1} + \mathcal{R}}{\int \left( \int F(t_1, \dots, t_k) dt_1 \right)^2 dt_2 \dots dt_k} \\
& =: \frac{I' + \mathcal{R}}{I}.
\end{aligned}$$

In the second line we removed the integration over the region  $\Gamma := \text{supp}(F) \setminus \Delta_k(0, \tau)$  at the cost of an error term

$$\mathcal{R} \leq \text{vol}(\Gamma) \sup_{x \in \Gamma} F(x)^2.$$

Now let  $\eta > 0$  be a small constant (to be chosen later) and introduce the function

$$(3.3) \quad F(t_1, \dots, t_k) := \begin{cases} \prod_{i=1}^k g(kt_i) & \text{for } (t_1, \dots, t_k) \in \Delta_k(2\eta, \tau) \\ \phi(t_1, \dots, t_k) & \text{for } (t_1, \dots, t_k) \in \Delta_k(\eta, \tau + \eta) \setminus \Delta_k(2\eta, \tau) \\ 0 & \text{otherwise} \end{cases}$$

with  $g$  taking the form

$$(3.4) \quad g(t) = 1_{[0, T]}(t)/(l + At).$$

Here  $l > 1$  and  $\tau = l^{-1}$ . This is a smooth modification of the function given in [5, Section 7]. Our intention is to choose a bump function  $\phi$  which makes  $F$  smooth and satisfies the bound

$$\max_{x \in \Delta_k(\eta, \tau + \eta) \setminus \Delta_k(2\eta, \tau)} \phi(x) \leq \max_{x \in (2\eta, \tau)} F(x).$$

For the construction of such a bump function it suffices to use a  $C^\infty$  version of Urysohn's lemma (see [3, 8.18]). We then gain control over  $\mathcal{R}$  by choosing  $\eta(k) > 0$  to be sufficiently small.

Before proceeding with the evaluation of  $I'$  and  $I$ , we note the estimates

$$\left( \int g(x) dx \right)^2 = \frac{\log \left( 1 + \frac{AT}{l} \right)^2}{A^2} \quad \text{and} \quad \nu := \int g(x)^2 dx = \frac{1}{lA} \left( 1 - \left( 1 + \frac{AT}{l} \right)^{-1} \right).$$

Due to the presence of the factor  $(1 - \sum_{i=2}^{k-1} t_i)^{-2}$  in the integral  $I'$  it will be convenient to assume that  $g$ 's center of mass is much smaller than 1. We impose the condition

$$(3.5) \quad m_c := \frac{\int xg(x)^2 dx}{\int g(x)^2 dx} \leq \frac{1}{l} \left( 1 - \frac{T}{k} \right).$$

**3.1. Estimates for  $\tilde{M}_k$ ,  $I'$  and  $I$ .** Let us first prove the estimate (3.2) for  $\tilde{M}_k$ .

**Lemma 3.1.** *For  $g$  defined as above and  $\epsilon := (1 - T/k)/l$ , one has the estimates*

$$(3.6) \quad \int_{\sum_{i=1}^k t_i \geq \epsilon} \left( \prod_{i=1}^k g(kt_i) \right)^2 dt_1 \dots dt_k \leq k^{-k} \nu^k \frac{T}{kl} \left( \frac{1 - \frac{T}{k}}{l} - m_c \right)^{-2}$$

and

(3.7)

$$\int_{\sum_{i=2}^k t_i \geq \epsilon} \left( \int \prod_{i=1}^k g(kt_i) dt_1 \right)^2 dt_2 \dots dt_k \leq k^{-(k+1)} \nu^{k-1} \frac{T}{kl} \left( \frac{1 - \frac{T}{k}}{l} - m_c \right)^{-2} \left( \int g(x) dx \right)^2.$$

*Proof.* Let  $\rho := (k - T)/l(k - 1) - m_c > 0$  and write  $x_i = kt_i$  for  $i = 1, \dots, k$ . We proceed as in [5, Section 7] and observe that the condition  $\sum_{i=1}^k t_i \geq \epsilon$  implies  $\sum_{i=1}^k x_i > (k - 1)m_c$ . This gives the inequality  $1 \leq \rho^{-2} \left( (k - 1)^{-1} \sum_{i=1}^k x_i - m_c \right)^2$  from which we gather that

$$\int_{\sum_{i=1}^k t_i \geq \epsilon} \left( \prod_{i=1}^k g(kt_i) \right)^2 dt_1 dt_2 \dots dt_k \leq \rho^{-2} k^{-k} \int \dots \int \left( \frac{1}{k-1} \sum_{i=1}^k x_i - m_c \right)^2 \left( \prod_{i=1}^k g(x_i) \right)^2 dx_1 \dots dx_k.$$

After expanding the square, a straightforward computation shows that the RHS is no greater than  $(\rho^{-2} k^{-k} m_c T \nu^k)/(k - 1)$ . The inequality (3.6) now follows since  $(k - 1)\rho^2 \geq k((1 - T/k)/l - m_c)^2$  and  $m_c \leq 1/l$ . A small modification of this argument gives (3.7).  $\square$

For the remainder of this section we impose the restrictions

$$(3.8) \quad \frac{T}{kl} \left( \frac{1 - \frac{T}{k}}{l} - m_c \right)^{-2} = o_k(1) \quad \text{and} \quad \log \left( 1 + \frac{AT}{l^2} \right) \sim \log k.$$

To prove (3.2) we first observe that  $\int_{\sum_{i=1}^k t_i \geq \tau} F(t_1, \dots, t_k)^2 = \int_{\sum_{i=1}^k t_i \geq \tau} \phi(t_1, \dots, t_k)^2$  can be controlled, as in the previous section, by taking  $\eta$  sufficiently small with respect to  $k$ . Combining (3.6) with the first estimate in (3.8) we now get

$$\int_{\sum_{i=1}^k t_i \leq \tau} F(t_1, \dots, t_k)^2 dt_1 \dots dt_k = \int_{\sum_{i=1}^k t_i \leq \tau} \left( \prod_{i=1}^k g(kt_i) \right)^2 dt_1 \dots dt_k = k^{-k} \nu^k (1 + o_k(1)).$$

On the other hand (3.7), together with the estimates in (3.8), yields

$$\begin{aligned} \int \left( \int F(t_1, \dots, t_k) dt_1 \right)^2 dt_2 \dots dt_k &\sim \int_{\sum_{i=2}^k t_i \leq \epsilon} \left( \int_0^{\tau - \sum_{i=2}^k t_i} \prod_{i=1}^k g(kt_i) dt_1 \right)^2 dt_2 \dots dt_k \\ &\sim \int_{\sum_{i=2}^k t_i \leq \epsilon} \left( \int \prod_{i=1}^k g(kt_i) dt_1 \right)^2 dt_2 \dots dt_k \\ &= k^{-(k+1)} \nu^{k-1} \left( \int g(x) dx \right)^2 (1 + o_k(1)). \end{aligned}$$

Combining all of the preceding estimates, we find that

$$\frac{J_k}{I_k} \sim \frac{l(\log k)^2}{kA} \left( 1 - \frac{T}{kl} \left( \frac{1 - \frac{T}{k}}{l} - m_c \right)^{-2} \right)$$

In order to find an appropriate choice of parameters  $A, T, l$ , we set  $1 + AT/l = e^\alpha$  with  $\alpha = \log k - c \log \log k$  for some constant  $c > 0$ . After a simple calculation one arrives at the expression

$m_c = (l/A)[\alpha/(1 - e^{-\alpha}) - 1]$ . Since  $T \leq le^\alpha/A = lk/(A(\log k)^c)$  it follows that

$$\frac{T}{kl} \left( \frac{1 - \frac{T}{k}}{l} - m_c \right)^{-2} \leq \frac{1}{A(\log k)^c} \left( \frac{1}{l} - \frac{1}{A(\log k)^c} - \frac{l(\alpha - 1)}{A} + O\left(\frac{l(\log k)^c}{Ak}\right) \right)^{-2}.$$

Choosing  $A = l^2(\log k)$  and  $l = \psi(k)^{-1}$ , the conditions in (3.5) and (3.8) are all met and we obtain (3.2) as well as the first estimate in Proposition 2.2.

Turning our attention to the integral  $I'$ , we find that

$$\begin{aligned} I' &\sim \int_{\sum_{i=2}^{k-1} t_i \leq \tau} \left( \int \int \prod_{i=2}^{k-1} g(kt_i) dt_1 \dots dt_k \right)^2 \left( 1 - \sum_{i=2}^{k-1} t_i \right)^{-2} dt_2 \dots dt_{k-1} \\ &\sim \int_{\sum_{i=2}^{k-1} t_i \leq \epsilon} \left( \prod_{i=2}^{k-1} g(kt_i)^2 \right) \left( \int g(kt) dt \right)^4 dt_2 \dots dt_{k-1} \\ &= \left( \frac{\log k}{kA} \right)^4 \nu^{k-2} (1 + o_k(1)). \end{aligned}$$

A similar computation shows that

$$I \sim \int_{\sum_{i=2}^k t_i \leq \epsilon} \left( \prod_{i=2}^k g(kt_i)^2 \right) \left( \int g(kt) dt \right)^2 dt_2 \dots dt_{k-1} = \left( \frac{\log k}{kA} \right)^2 \nu^{k-1} (1 + o_k(1)).$$

We conclude that  $I'/I = (1 + o_k(1))\psi(k)\frac{\log k}{k}$  which gives the second part of Proposition 2.2.

#### 4. GOLDBACH VERSUS POLIGNAC NUMBERS

In this section we prove Theorem 1.1 by combining a Cauchy-Schwarz argument with a result from graph theory.

Let  $k$  be a large natural number and suppose  $\mathcal{H} = \{h_1, \dots, h_k\}$  is an admissible  $k$ -tuple. Assume furthermore that  $N \in \mathbb{N}$  is sufficiently large with respect to  $k$ . Associated to  $\mathcal{H}$  there is another tuple  $\mathcal{H}' = \{h'_1, \dots, h'_k\}$ , where  $h'_j = N - h_j$  for each  $j$ . Now write  $\overline{\mathcal{H}} = \mathcal{H} \cup \mathcal{H}'$  and define

$$\tilde{S}(\overline{\mathcal{H}}) = \sum_{n \in [N/2, N]} \left( \sum_{h \in \overline{\mathcal{H}}} a_h(n) \right) w(n)^2.$$

where  $a_h(n) = 1_{\mathcal{P}}(n + h)$  when  $h \in \mathcal{H}$  and  $a_h(n) = 1_{\mathcal{P}}(h - n)$  when  $h \in \mathcal{H}'$ . An application of Cauchy-Schwarz gives

$$(4.1) \quad \tilde{S}(\overline{\mathcal{H}}) \leq \left( \sum_{n \in [N/2, N]} 1_{\{X > 0\}}(n) w(n)^2 \right)^{1/2} \left( \sum_{n \in [N/2, N]} \sum_{\substack{h, \tilde{h} \in \overline{\mathcal{H}} \\ h \neq \tilde{h}}} a_h(n) a_{\tilde{h}}(n) w(n)^2 \right)^{1/2}$$

where  $X(n) = X_{\overline{\mathcal{H}}}(n) = \sum_{h \in \overline{\mathcal{H}}} a_h(n)$ . From (4.1) and Proposition 2.2. we get the lower bound

$$\begin{aligned}
\sum_{n \in [\frac{N}{2}, N]} w(n)^2 &\geq \sum_{n \in [N/2, N]} 1_{\{X > 0\}}(n) w(n)^2 \\
&\geq \left( \sum_{n \in [N/2, N]} \sum_{h \in \overline{\mathcal{H}}} a_h(n) w(n)^2 \right)^2 \left( \sum_{n \in [\frac{N}{2}, N]} \sum_{\substack{h, \tilde{h} \in \overline{\mathcal{H}} \\ h \neq \tilde{h}}} a_h(n) a_{\tilde{h}}(n) w(n)^2 \right)^{-1} \\
&= \left[ \delta \frac{N}{2} \beta(N) \psi(2k) \log(2k) I_{2k}(f) \right]^2 \left( \sum_{n \in [\frac{N}{2}, N]} \sum_{\substack{h, \tilde{h} \in \overline{\mathcal{H}} \\ h \neq \tilde{h}}} a_h(n) a_{\tilde{h}}(n) w(n)^2 \right)^{-1} (1 + o_k(1)).
\end{aligned}$$

Now let  $\mathcal{M} := (\mathcal{H} \times \mathcal{H}) \cup (\mathcal{H}' \times \mathcal{H}')$  and write  $\mathcal{M}^c$  for the complement of  $\mathcal{M}$  in  $\overline{\mathcal{H}} \times \overline{\mathcal{H}}$ . We gather that

$$\begin{aligned}
(4.2) \quad \sum_{n \in [N/2, N]} \sum_{\substack{(h, \tilde{h}) \in \mathcal{M}^c \\ h \neq \tilde{h}}} a_h(n) a_{\tilde{h}}(n) w(n)^2 &\geq \frac{(\delta(N/2) \beta(N) \psi(2k) \log(2k) I_{2k}(f))^2}{(N/2) \beta(N) I_{2k}(f)} (1 + o_k(1)) \\
&\quad - \sum_{n \in [N/2, N]} \sum_{\substack{(h, \tilde{h}) \in \mathcal{M} \\ h \neq \tilde{h}}} a_h(n) a_{\tilde{h}}(n) w(n)^2.
\end{aligned}$$

To prove Theorem 1.1 we set  $\delta = 1/4 + \epsilon$  and consider two mutually exclusive assumptions.

**Hypothesis A** We say hypothesis A holds if there exists an increasing sequence of natural numbers  $k$  satisfying the following condition. For each admissible  $k$ -tuple  $\mathcal{H}$  at least  $1/2 - \epsilon$  of all pairs  $1 \leq i < j \leq k$  produce a difference  $h_j - h_i$  which is not a de Polignac number.

Suppose hypothesis A is true and let  $n \in [N/2, N]$ . It follows that  $a_h(n) a_{\tilde{h}}(n) = 0$  for at least  $1/2 - \epsilon$  of all pairs  $(h, \tilde{h}) \in \mathcal{M}$ . Plugging this information back into (4.2) and applying Proposition 2.2 we find that

$$\begin{aligned}
\sum_{n \in [N/2, N]} \sum_{\substack{(h, \tilde{h}) \in \mathcal{M}^c \\ h \neq \tilde{h}}} a_h(n) a_{\tilde{h}}(n) w(n)^2 &\geq \delta^2 \frac{N}{2} \beta(N) (\psi(2k))^2 (\log 2k)^2 I_{2k}(f) (1 + o_k(1)) \\
&\quad - 2(1/2 + \epsilon) \delta \frac{N}{2} \beta(N) \psi(2k)^2 \frac{(\log 2k)^2}{4k^2} (k^2 - k) I_{2k}(f) (1 + o_k(1)).
\end{aligned}$$

A simple calculation shows that the RHS is a positive quantity for  $N$  and  $k$  sufficiently large. From this we deduce the existence of some  $n \in [N/2, N]$  and a pair  $h_i, h_j \in \mathcal{H}$  for which  $n + h_i$  and  $N - n - h_j$  are both prime. This implies that all sufficiently large  $N$  lie within a bounded distance from a Goldbach number.

Now consider the case where hypothesis A fails and write  $\mathcal{D}$  for the set of de Polignac numbers. Let  $k$  be any sufficiently large number and  $\mathcal{H}$  an admissible  $k$ -tuple. Then at least  $1/2 + \epsilon$  of all pairs  $1 \leq i < j \leq k$  give a difference  $h_j - h_i$  which is a de Polignac number. As an immediate consequence we get the following useful property. Let  $U := \{u_1, \dots, u_k\} \subset 2\mathbb{N}$  and  $V := \{v_1, \dots, v_k\} \subset 2\mathbb{N}$  be a



pair of sets for which  $U \cap V = \emptyset$  and  $U \cup V$  is admissible. Then there exists a  $(u, v) \in U \times V$  with  $|u - v| \in \mathcal{D}$ . We will say  $(\mathcal{D}, k)$  satisfies the cross product property. To finish the proof of Theorem 1.1 we need the following lemma which was proven in a private communication with S. Miner and S. Das.

**Lemma 4.1.** *Let  $k \in \mathbb{N}$  be arbitrary and suppose  $(\mathcal{D}, k)$  satisfies the cross product property. Then  $\mathcal{D}$  has full asymptotic density in  $2\mathbb{N}$ . Moreover, we have the power saving*

$$(4.3) \quad |\mathcal{D}^c \cap [N]| \ll N^\kappa$$

for some  $\kappa < 1$  depending on  $k$ .

**Remark** There is an expedient way of establishing the full density of  $\mathcal{D}$  without the power saving result. Indeed, suppose for a contradiction that the set  $A := \mathcal{D}^c \cap 2\mathbb{N}$  has positive upper density and let  $\mathcal{P}(y) = \prod_{p \leq y} p$ . An application of Szemerédi's Theorem [6] gives a  $(2k-2)$ -term arithmetic progression  $P = (b+ra)_{r \leq 2k-2} \subset A$ . We may assume without loss of generality that  $a \equiv 0 \pmod{\mathcal{P}(2k)}$ . Now consider the pair  $U = \{a, 2a, \dots, ka\}$  and  $V = \{b+ka, b+(k+1)a, \dots, b+(2k-1)a\}$ . Clearly  $U$  and  $V$  do not intersect and their union is admissible. Since the difference set  $|U - V| = P \subset A$ , the cross product property gives the desired contradiction.

We now turn to the estimate for  $\mathcal{D}^c \cap [N]$ . The result will follow from two simple lemmas. Call a pair  $\{x, y\}$  an  $A$ -pair if  $|y - x| \in A$ .

**Lemma 4.2.** *For every  $k$  there is an  $\ell = \ell(k)$  such that if  $U$  and  $T$  are two disjoint subsets of  $2[N]$  of size  $\ell$ , then there are  $X \subset U$  and  $Y \subset T$  such that  $|X| = |Y| = k$  and  $X \cup Y$  is admissible.*

Given the above lemma, we shall use the classic result of Kővári–Sós–Turán [4] on the Turán number of complete bipartite graphs to resolve the problem.

**Theorem 4.3** (Kővári–Sós–Turán, 1954). *If  $G$  is a graph on  $n$  vertices that does not contain  $K_{t,t}$  as a subgraph, then  $G$  has at most  $ct^{1/t}n^{2-1/t} + O(n)$  edges.*

*Proof of Lemma 4.1.* Let  $H$  be a graph with vertices  $V = 2[N]$ , and edges

$$E = \{\{x, y\} : \{x, y\} \text{ is not an } A\text{-pair}\}.$$

Let  $(\mathcal{D}, k)$  satisfy the cross product property and let  $\ell = \ell(k)$  be as in Lemma 4.2. We claim that  $H$  is  $K_{\ell, \ell}$ -free. Indeed, suppose for contradiction  $K_{\ell, \ell} \subset H$ , and let  $U$  and  $T$  be the two vertex sets on which this copy of  $K_{\ell, \ell}$  is realised. In particular, we must have  $U \times T \subset E(H)$ , and so there are no  $A$ -pairs in  $U \times T$ .

However, by Lemma 4.2, we can find two  $k$ -sets  $X \subset U$  and  $Y \subset T$  such that  $X \cup Y$  is admissible. By assumption there must be some  $A$ -pair in  $X \times Y \subset U \times T$ , giving the necessary contradiction. Thus  $H$  is indeed  $K_{\ell, \ell}$ -free, and by Theorem 4.3 has  $O(N^{2-1/\ell})$  edges. However, since there are  $N - d$  edges corresponding to a difference of  $2d$ , we need at least  $\binom{t+1}{2}$  edges to cover  $t$  differences. Since  $H$  has only  $O(N^{2-1/\ell})$  edges, it can cover at most  $O(N^{1-1/(2\ell)})$  differences, and thus we must have  $|A| \geq N - O(N^{1-1/(2\ell)})$ .  $\square$

It remains to prove Lemma 4.2.

*Proof of Lemma 4.2.* Since we wish to find sets  $X$  and  $Y$  of size  $k$ , we need only consider primes of size at most  $2k$ . Since  $U, T \subset 2[N]$  consist solely of even integers, we need only take into account odd primes  $p_2 < \dots < p_m \leq 2k$ , where  $m = \pi(2k)$ . To begin, set  $X_1 = U$  and  $Y_1 = T$  and  $\ell_0 = \ell = 3^m k$ .

Now suppose for  $2 \leq i \leq m-1$  we are given subsets  $X_i \subset U$  and  $Y_i \subset T$ , both of size  $\ell_i$ , such that  $X_i \cup Y_i$  does not occupy all residue classes modulo  $p_j$  for any  $2 \leq j \leq i$ . By the pigeonhole principle, there is some residue class  $C$  modulo  $p_{i+1}$  such that  $|(X_i \cup Y_i) \cap C| \leq 2\ell_i/p_{i+1}$ . Let  $X'_{i+1} = X_i \setminus C$  and  $Y'_{i+1} = Y_i \setminus C$ . Let  $\ell_{i+1} = \ell_i(1 - 2/p_{i+1})$ , and observe that this gives a lower bound on the sizes of  $X'_{i+1}$  and  $Y'_{i+1}$ . Finally, take  $X_{i+1}$  and  $Y_{i+1}$  to be arbitrary subsets of  $X'_{i+1}$  and  $Y'_{i+1}$  of size  $\ell_{i+1}$ , and note that these sets do not occupy the residue class  $C$  modulo  $p_{i+1}$ .

Repeating these process, we arrive at sets  $X_m \subset U$  and  $Y_m \subset T$  of size  $\ell_m$  that do not occupy all residue classes modulo  $p_j$  for any  $1 \leq j \leq m$ , and hence are admissible. If  $\ell_m \geq k$ , we can take  $X$  and  $Y$  to be arbitrary  $k$ -subsets of  $X_m$  and  $Y_m$ .

We have  $\ell_m = \ell_{m-1}(1 - 2/p_m) = \dots = \ell \prod_{j=2}^m (1 - 2/p_j) \geq \ell/3^m = k$ , completing the proof.  $\square$

## 5. A NOTE ON THE SEQUENCE OF NORMALISED PRIME GAPS

In this section we give a proof of Proposition 1.2 and discuss a conditional improvement of the result which relies on a conjectural form of [1, Theorem 4.2] combined with a simple Cauchy Schwarz estimate. We first require some notation and background results. Recall [1, Lemma 4.1].

**Lemma 5.1.** *Let  $T \geq 3$  and assume  $P \geq T^{1/\log_2 T}$ . Then there exists an absolute constant with the following property. Ranging over all moduli  $q$  satisfying  $q \leq T$  and  $P^+(q) \leq P$  there is at most one primitive character  $\chi \bmod q$  for which  $L(s, \chi)$  has a zero in the region*

$$\operatorname{Re}(s) \geq 1 - \frac{c}{\log P}, \quad |\operatorname{Im}(s)| \leq \exp[\log P / (\log T)^{1/2}].$$

In this case, one has the bounds

$$P^+(q) \gg \log q \gg \log_2 T.$$

Following the notation of Lemma 5.1 we introduce the quantities

$$Z(N^T) = P^+(q), \quad w = \epsilon \log N, \quad W = \prod_{\substack{p \leq w \\ p \nmid Z(N^{4\epsilon})}} p,$$

and consider a modified form of the Bombieri-Vinogradov theorem. For squarefree  $q_0$  satisfying  $P^+(q_0) \leq N^{\epsilon/\log_2 N}$  we require an estimate of the following form. There exists a constant  $0 < \theta < 1$  so that for any small  $\delta > 0$

$$(5.1) \quad \sum_{\substack{q \leq N^{\theta-\delta} \\ q_0 | q \\ (q, Z_{N^{2\epsilon}}) = 1}} \max_{(a, q) = 1} \left| \psi(N; q, a) - \frac{N}{\varphi(q)} \right| \ll_{A, \delta} \frac{N}{\varphi(q_0)(\log N)^A}.$$

In [1, Theorem 4.2] it was demonstrated that (5.1) holds with  $\theta = 1/2$ .

Now assume  $k \in \mathbb{N}$  is large and let  $\mathcal{H} = \{h_1, \dots, h_k\}$  be an admissible  $k$ -tuple for which each member is bounded in size by  $N$ . Assume also that each prime dividing  $\prod_{1 \leq i < j \leq k} (h_i - h_j)$  is smaller than  $w$ .

In our current setting we require a modified version of the weight (2.2). For a  $k$ -tuple  $\underline{d} = (d_1, \dots, d_k)$ , define

$$\lambda_{\underline{d}} = \left( \prod_{i=1}^k \mu(d_i) \right) \sum_{j=1}^J \prod_{l=1}^k F_{j,l} \left( \frac{\log d_l}{\log N} \right)$$

with  $J$  a fixed number,  $F_{j,l} : [0, \infty] \rightarrow \mathbb{R}$  smooth and compactly supported. We also assume  $\lambda_{\underline{d}}$  is supported on  $k$ -tuples for which  $((\prod_{i=1}^k d_i), Z(N^{4\epsilon})) = 1$  and  $(\prod_{i=1}^k d_i) \leq N^\delta$ . We let  $\nu$  denote the associated weight function given in (2.2).

**Proposition 5.2.** *Let  $a = \lceil 2/\theta \rceil$ ,  $m \in \mathbb{N}$  and  $k$  a large natural number with  $am+1 \mid k$ . Suppose, in addition to all of the above assumptions, that the  $k$ -tuple  $\mathcal{H}$  is partitioned into two parts*

$$\mathcal{H} = \mathcal{H}_1 \cup \dots \cup \mathcal{H}_{am+1}$$

*with  $|\mathcal{H}_i^{(1)}| = k/(am+1)$  for each  $i$ . Then there exists an  $n_1 \in [N, 2N]$  for which  $n_1 \equiv b \pmod{W}$  together with a set of  $m+1$  distinct indices  $\{i_1^{(1)}, \dots, i_{m+1}^{(1)}\} \subset \{1, \dots, am+1\}$  satisfying*

$$|\mathcal{H}_i^{(1)} \cap \mathcal{P}| = 1 \text{ for all } i \in \{i_1, \dots, i_{m+1}\}$$

*Proof.* Define  $Y_j(n) = \sum_{h \in \mathcal{H}_j} 1_{\mathcal{P}}(n+h)$  for each  $1 \leq j \leq am+1$  and consider the sum

$$A = \sum_{n \leq N} \left( \sum_{j=1}^{am+1} 1_{Y_j > 0}(n) - m - \sum_{j=1}^{am+1} \sum_{h, h' \in \mathcal{H}_j} 1_{\mathcal{P}}(n+h) 1_{\mathcal{P}}(n+h') \right) \nu(n)^2.$$

Observe that the result will follow if we are able to demonstrate that  $A > 0$ . By choosing the functions  $F_{j,l}$  appropriately the following bounds were proven in [1, Lemma 4.5 parts (i), (ii), (iii)]. For any  $0 < \rho < 1$  and any small  $\delta > 0$  one has

$$\begin{aligned} (5.2) \quad & \sum_{n \leq N} \nu(n)^2 \sim N\beta(N)I_k(F)(1 + o_k(1)) \\ & \sum_{n \leq N} 1_{\mathcal{P}}(n+h)\nu(n)^2 \sim N\beta(N)\frac{\log k}{k}(\rho\delta)I_k(F)(1 + o_k(1)) \\ & \sum_{n \leq N} 1_{\mathcal{P}}(n+h)1_{\mathcal{P}}(n+h')\nu(n)^2 \leq N\beta(N)\left(\frac{2}{\theta} + O(\delta)\right)\frac{(\log k)^2}{k^2}(\rho\delta)^2 I_k(F)(1 + o_k(1)), \end{aligned}$$

where  $F(y_1, \dots, y_k) := \sum_{j=1}^J \prod_{l=1}^k F'_{j,l}(y_l)$ . We address the first summation in  $A$  with a Cauchy-Schwarz argument. Writing  $\rho\delta \log k = cm$  for some small constants  $\delta, c > 0$  it follows that

$$\begin{aligned} (5.3) \quad & \sum_{n \leq N} 1_{\{Y_j > 0\}}(n)\nu(n)^2 \geq \left( \sum_{n \leq N} \sum_{h \in \mathcal{H}_j} 1_{\mathcal{P}}(n+h)\nu(n)^2 \right)^2 \left( \sum_{n \leq N} \sum_{\substack{h, h' \in \mathcal{H}_j \\ h \neq h'}} 1_{\mathcal{P}}(n+h)1_{\mathcal{P}}(n+h')\nu(n)^2 \right)^{-1} \\ & \geq \left[ (\rho\delta)N\beta(N)\frac{\log k}{k}\frac{k}{am+1} I_k(F) \right]^2 (1 + o_k(1)) \\ & \times \left[ \frac{2}{\theta}(1 + O(\delta))(\rho\delta)^2 N\beta(N)\frac{(\log k)^2}{k^2} \left( \left( \frac{k}{am+1} \right)^2 - \frac{k}{am+1} \right) I_k(F) \right]^{-1} \\ & \geq N\beta(N)I_k(F)\frac{\theta}{2}(1 + o_k(1))(1 + O(\delta)). \end{aligned}$$

Applying the bounds given in (5.2) we get

$$A \geq N\beta(N)I_k(F)(1+o_k(1))(1+O(\delta)) \left( \frac{\theta}{2}(am+1) - m - (am+1) \left( \left( \frac{k}{am+1} \right)^2 - \frac{k}{am+1} \right) \frac{c^2 m^2}{k^2} \right).$$

Since  $a \geq (2/\theta)$ , the result follows after taking  $c$  to be sufficiently small and  $k$  sufficiently large.  $\square$

From this point onwards the demonstration of Proposition 1.2 is carried out as in [1, Section 6]. Let  $m \geq 1$  and suppose  $k$  is a large positive integer with  $(am+1)|k$ . Given  $\beta_{am+1} \geq \dots \geq \beta_1 > 0$  one obtains a  $k$ -tuple  $\mathcal{H} = \mathcal{H}_1 \cup \dots \cup \mathcal{H}_{am+1}$  for which each set in the partition is of size  $k/(am+1)$  and

$$h_j = (\beta_j + \epsilon + o(1)) \log N \text{ for all } h_j \in \mathcal{H}_j.$$

Furthermore, one finds an integer  $n > y$  and  $z > 0$  so that  $[n, n+z] \cap \mathcal{P} = \mathcal{H}(n) \cap \mathcal{P}$ . We gather that the primes in  $\mathcal{H}(n)$  are consecutive. By Proposition 5.2 there are at least  $m+1$  primes, each coming from a distinct member of the partition. In this manner we obtain a string of indices  $1 \leq i_1 < i_2 < \dots < i_l \leq am+1$ , coming from distinct cells in the partition, with  $l \geq m+1$  and associated representations

$$\frac{p_{r+1} - p_r}{\log p_r} = \beta_{i_{j+1}} - \beta_{i_j} + o(1)$$

for some value  $r = r(i_j)$ . From this we easily deduce the following property.

(5.4)

Any sequence  $0 \leq \beta_1 < \dots < \beta_{am+1}$  contains a subset  $\beta_{i_1} < \beta_{i_2} < \dots < \beta_{i_l}$  of length  $l \geq m+1$  satisfying  $\beta_{i_{j+1}} - \beta_{i_j} \in \mathcal{L} \forall i_j$ .

**Lemma 5.3.** *Property (5.4) implies Proposition 1.2. Assuming a level of distribution  $\theta = 1$  in (5.1), one gets the improvement*

$$\liminf_{T \rightarrow \infty} \frac{m([0, T] \cap \mathcal{L})}{T} \geq \frac{1}{2}.$$

*Proof.* The unconditional claim is an application of [1, Corollary 1.2] for the case  $m = 1$ ,  $\theta = 1/2$ .  $\square$

## APPENDIX

In this final section we will discuss Proposition 2.1. Since the proof follows that of [2, Lemma 4.1] very closely, we will limit ourselves to a sketch of the argument, pointing out important differences when necessary. Expanding the expression  $S$  in (2.1) we get two sums. First consider

(A-1)

$$\begin{aligned} \sum_{n \leq N} w(n)^2 &= \sum_{\underline{d}, \underline{e}} \left( \prod_{i=1}^k \mu(d_i) \mu(e_i) \right) f\left(\frac{\log d_1}{\log x}, \dots, \frac{\log d_k}{\log x}\right) f\left(\frac{\log e_1}{\log x}, \dots, \frac{\log e_k}{\log x}\right) \sum_{\substack{n \leq N \\ n \equiv b \pmod{W} \\ n + h_i \equiv 0 \pmod{[d_i, e_i]}}} 1 \\ &= \sum_{\underline{d}, \underline{e}} \left( \prod_{i=1}^k \mu(d_i) \mu(e_i) \right) f\left(\frac{\log d_1}{\log x}, \dots, \frac{\log d_k}{\log x}\right) f\left(\frac{\log e_1}{\log x}, \dots, \frac{\log e_k}{\log x}\right) \left( \frac{N}{W \prod_{j=1}^k [d_j, e_j]} + O(1) \right). \end{aligned}$$

Since  $f$  is a compactly supported, smooth function, we may apply Fourier inversion to write

$$(A-2) \quad \exp\left(\sum_{i=1}^k t_i\right) f(\underline{t}) = \int_{\mathbb{R}^k} \exp\left(-i \sum_{i=1}^k t_i \xi_i\right) g(\xi) d\xi \quad \forall \underline{t} = (t_1, \dots, t_k) \in \mathbb{R}^k.$$

for some smooth function  $g : \mathbb{R}^k \rightarrow \mathbb{R}$  obeying decay estimates of the form  $g(\xi) \ll_A (1 + \|\xi\|)^{-A}$  for any  $A > 0$ . This leads to the expression

$$f\left(\frac{\log d_1}{\log x}, \dots, \frac{\log d_k}{\log x}\right) = \int_{\mathbb{R}^k} \frac{g(\xi)}{\prod_{i=1}^k d_i^{\frac{1+i\xi_i}{\log x}}} d\xi.$$

Inserting this integral representation into the main term of (A-1) we find the sum

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^k} K(\xi, \xi') g(\xi) g(\xi') d\xi d\xi',$$

where

$$K(\xi, \xi') = \sum_{\underline{d}, \underline{e}}^* \prod_{i=1}^k \frac{\mu(d_i) \mu(e_i)}{[d_i, e_i] d_i^{\frac{1+i\xi_i}{\log x}} e_i^{\frac{1+i\xi'_i}{\log x}}}.$$

The superscript  $\star$  means the summation takes place over squarefree integers for which  $[d_1, e_1], \dots, [d_k, e_k], W, Q$  are pairwise coprime. In [2, Equation 41] the asymptotic for  $K$  was shown to be

$$K(\xi, \xi') = (1 + o(1)) \beta(N) N \prod_{j=1}^k \frac{(1 + i\xi_j)(1 + i\xi'_j)}{2 + \xi_j + i\xi'_j}.$$

To prove the identity

$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \prod_{j=1}^k \frac{(1 + i\xi_j)(1 + i\xi'_j)}{2 + i\xi_j + i\xi'_j} g(\xi) g(\xi') d\xi d\xi' = \int_{\mathbb{R}_+^k} f(t)^2 dt$$

divide the RHS of (A-2) by  $\exp(\sum_{i=1}^k t_i)$  and differentiate the integrand with respect to each variable  $t_i$ . This gives

$$f(t) = \int_{\mathbb{R}^k} \prod_{j=1}^k (1 + i\xi_j) \exp\left(-\sum_{r=1}^k t_r (1 + i\xi_r)\right) g(\xi) d\xi$$

which is then squared and integrated to get the desired representation.

It remains to evaluate the summation over primes appearing in (2.1). For any index  $1 \leq r \leq k$  one expands the sum

$$\sum'_{n \leq N} 1_{\mathcal{P}}(n + h_r) w(n)^2$$

to find the expression

(A-3)

$$\begin{aligned}
& \sum_{\underline{d}, \underline{e}} \left( \prod_{i=1}^k \mu(d_i) \mu(e_i) \right) f \left( \frac{\log d_1}{\log x}, \dots, \frac{\log d_k}{\log x} \right) f \left( \frac{\log e_1}{\log x}, \dots, \frac{\log e_k}{\log x} \right) \sum_{\substack{n \leq N \\ n \equiv b \pmod{W} \\ n+h_i \equiv 0 \pmod{[d_i, e_i]}}} 1_{\mathcal{P}}(n+h_r) \\
&= \sum_{\underline{d}, \underline{e}} \left( \prod_{i=1}^k \mu(d_i) \mu(e_i) \right) f \left( \frac{\log d_1}{\log x}, \dots, \frac{\log d_k}{\log x} \right) f \left( \frac{\log e_1}{\log x}, \dots, \frac{\log e_k}{\log x} \right) \\
&\quad \times \left[ \frac{N}{\varphi(q(W, \underline{d}, \underline{e})) \log N} + \Delta \left( 1_{[N+h_r, N+2h_r]} \theta, a(W, \underline{d}, \underline{e}), q(W, \underline{d}, \underline{e}) \right) \right].
\end{aligned}$$

Here we have used the notation  $q(W, \underline{d}, \underline{e}) = W \prod_{j=1}^k [d_j, e_j]$  and  $a(W, \underline{d}, \underline{e})$  is the unique residue class mod  $q(W, \underline{d}, \underline{e})$  satisfying all the conditions on  $n$ . The main term in (A-3) is treated as before except that the factors  $[d_i, e_i]$  in the denominator become  $\varphi([d_i, e_i])$ , which will have no effect on the argument.

For the remainder term  $\Delta$ , we refer the reader to [2, Section 4.3], where it is demonstrated that a bound of the form

$$\sum_{\substack{q \leq x^{1/2+2\varpi} \\ P^+(q) \ll x^{2\sigma}}} \Delta \left( 1_{[h_r, N+h_r]} \theta, a, q, \right) \ll_A \frac{N}{(\log N)^A}$$

holds for some pair of constants  $\varpi, \sigma > 0$ . We observe that the smoothness parameter  $\sigma$ , which plays an important role in the above equidistribution estimate, forces the support of  $f$  to lie within  $\Delta_k(0, \sigma)$ . This accounts for the occurrence of  $\sigma$  in Propositions 2.1 and 2.2.

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